THE JOURNAL

OF THE

Indian Mathematical Society

Vol. XIII.] FEBRUARY 1921. [No. 1

AN ALGEBRA OF ARITHMETIC FUNCTIONS.

BY F. HALLBERG.

(*Continued from Vol. XII, page 216.*)

The Units of the Algebra.

§ 17. *Definitions.*

From the development so far, it is evident that our algebra may be considered as a quasi-linear algebra, differing from ordinary linear algebras in having an infinite number of units. Such an algebra may conveniently be called an *infinitary* algebra.

There is evidently a principal unit or modulus

$$e_1 = [1, 0, 0, 0, \ldots] = 1, \qquad \ldots \qquad \ldots \qquad (17.1)$$

the remaining units being defined by

$$\left.\begin{array}{l} e_2 = [0, 1, 0, 0, \ldots] \\ e_3 = [0, 0, 1, 0, \ldots] \\ e_4 = [0, 0, 0, 1, \ldots] \\ \ldots \quad \ldots \quad \ldots \quad \ldots \end{array}\right\} \qquad \ldots \qquad \ldots \qquad (17.11)$$

Hence every A. F., $[x_n]$, can be expressed in the form

$$X = [x_n] = \sum_{n=1}^{\infty} x_n e_n. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (17.2)$$

the elements of the A. F. being regarded as co-ordinates of an infinitary complex number X, and belonging to a given number field, with respect to which the units e_n are linearly independent.

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From the definition of the process of multiplication in formula (2 . 3), it follows that the units are subject to the multiplication table

$$e_m\, e_n = e_{mn}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (17.3)$$

so that the product of two infinitary complex numbers, X and X', may be written

$$XX' = X'X = \sum_{m,\, n=1}^{\infty} x_m\, x'_n\, e_{mn}, \quad \ldots \quad \ldots \qquad (17.31)$$

Thus the product is itself an infinitary complex number.

§ 18. *Some characteristic features of the infinitary algebra considered.*

Assume that for two infinitary numbers, X and X', with *real* co-ordinates, x_n and x'_n

$$XX' \equiv 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (18.1)$$

Excluding the trivial case in which both the given infinitary numbers vanish, we assume further that

$$x'_1 \equiv x'_2 \equiv x'_3 \equiv \ldots x'_{r-1} \equiv 0, \quad \ldots \quad \ldots \qquad (18.11)$$

$$x'_r \not\equiv 0, \quad \ldots \quad \ldots \quad \ldots \qquad (18.111)$$

r being a positive integer.

By (18.11), the co-ordinate with index r in the product is reduced to

$$x_1\, x'_r \equiv 0, \quad \ldots \quad \ldots \qquad (18.12)$$

from which follows, by (18.111)

$$x_1 \equiv 0. \quad \ldots \quad \ldots \qquad (18.121)$$

Next, let p be a prime, and consider the co-ordinate with index pr in the product. By the aid of (18.11), (18.121) we obtain

$$x_p\, x'_r \equiv 0, \quad \ldots \quad \ldots \qquad (18.13)$$

and hence, by (18.111)

$$x_p \equiv 0. \quad \ldots \quad \ldots \qquad (18.131)$$

If now q is another prime, different or not from p, the co-ordinate with index pqr in the product becomes, by (18.11), (18.121), (18.131)

$$x_{pq}\, x'_r \equiv 0, \quad \ldots \quad \ldots \qquad (18.14)$$

and again, by (18.111)

$$x_{pq} \equiv 0. \quad \ldots \quad \ldots \qquad (18.141)$$

Continuing in this way, we see that each co-ordinate x_n of X is identically zero. This result may be expressed as follows :—

The necessary and sufficient condition that a product of numbers with real co-ordinates in the infinitary algebra considered shall be zero, is that at least one factor is zero.

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Consequently the theorem of Frobenius, that the only linear associative algebras over the field of reals in which a product is zero only when one factor is zero, are the algebra of reals, the algebra of ordinary complex numbers, and the algebra of real quaternions, cannot be extended to infinitary algebras. (For the above theorem, with references, see L. E. Dickson, *Linear Algebra* (Cambr. Math. Tracts, No. 16), p. 10—12.)

It is clear that the above proof is valid also if the co-ordinates of our algebra are ordinary complex numbers.

It follows, that our algebra is not nilpotent, since no number X exists, such that

$$X \neq 0, \qquad \dots \qquad \dots \qquad (18.2)$$

$$X^k = 0, \qquad \dots \qquad \dots \qquad (18.21)$$

k being a positive integer.

One of the most interesting questions in connection with our algebra is that of the process of division. We noted already in § 2, that if the co-ordinate for the modulus in the divisor vanishes, while the corresponding co-ordinate in the dividend does not vanish, the operation is not possible. This followed at once from formula (8.9), from which we also deduced, that if the first mentioned co-ordinate did not vanish, the quotient always existed. But there are other possibilities. Thus

$$\frac{e_m}{e_n} = e_{\frac{m}{n}} \qquad \dots \qquad \dots \qquad (18.3)$$

provided m is a multiple of n; otherwise the quotient does not exist Similarly, from an identity such as

$$(ae_m + be_n)(\alpha e_r + \beta e_s) \equiv a\alpha e_{mr} + a\beta e_{ms} + b\alpha e_{nr} + b\beta e_{ns}, \quad \dots \quad (18.4)$$

it is easily seen under what conditions an expression like that on the right may be factorised. A discussion of the general case would carry us too far. It may be observed, however, that if e_m and e_n are the units with non-zero co-efficients of lowest order in the dividend and divisor respectively then the condition for the validity of (18.3) is a necessary (but not sufficient) condition for the existence of the quotient.

[It is perhaps worth mentioning that some of the above anomalies with respect to the operation of division may be avoided by the admission of units with *fractional* indices, subiect to the multiplication table (17.3). Since other complications are introduced simultaneously however, the algebra obtained will not be further considered here.]

In § 3, we utilised a connection between our algebra and the theory of ordinary Dirichlet's series to attain brevity and clearness in our representation. It will now be readily understood that the process indicated

furnishes the results obtained because of the fact, that we may, provisionally, put

$$e_n \sim n^{-s}, \quad \dots \quad \dots \quad (18.5)$$

where s is an *arbitrary* complex number, the function on the right behaving in a manner similar to that of the unit on the left with respect to the fundamental operations of ordinary algebra. In the place mentioned, we remarked that the convergence or non-convergence of the D. S. was of no account. This we now propose to prove.

For this purpose, assume the more general equivalence

$$e_n \sim \{f(n)\}^{-s}, \quad \dots \quad \dots \quad (18.6)$$

$f(n)$ satisfying the functional relation (5.1). This implies

$$f(1) = 1. \quad \dots \quad \dots \quad (18.61)$$

Let now an infinitary number X be given. It is clear, that a sequence of primes q_r may be chosen, corresponding to the natural sequence of all primes p_r, such that

$$f(p_r) = q_r, \quad \dots \quad \dots \quad (18.62)$$

and that the D. S.

$$\sum_{n=1}^{\infty} x_n \{f(n)\}^{-s} \quad \dots \quad \dots \quad (18.63)$$

possesses a half-plane of absolute convergence, the primes q_r being taken all different, and arranged in order of magnitude. It is further evident, that if some of the primes q_r are chosen larger, the new D. S. will still possess a half-plane of absolute convergence, with a limiting abscissa not larger than that of the former. If now a set of infinitary numbers be given, while others are derived from this set by means of functional operations, the sequence q_r may be so chosen, that every infinitary number corresponds to a D. S. of the form (18.63), with a half-plane of absolute convergence since any function of one or more such D. S. may be written in the form of a D. S. of the same type, $f(n)$ satisfying the relation (5.1). In the region common to all those half-planes the arbitrary s' is supposed to be situated. Then the terms may be rearranged so that the D.S. become of the ordinary type, with this peculiarity, that there is a set of zero coefficients of constant order common to all the D. S.

Let now two equivalent functional processes be applied to series of our set. The results must be identical, *i.e.*, the difference between the results must vanish identically. But then all the coefficients in this difference must be identically zero, otherwise we could find a value for s in the common half-plane of absolute convergence for which the D. S. in question has a sum different from zero. [Vide E. Landau *Handbuch der Lehre von der Verteilung der Primzahlen*, p. 747]. Passing over to our algebra, we have the important theorem :

Equivalent functional processes, applied to the same infinitary numbers of our algebra, furnish identical results.

This theorem justifies the definition of inverse functions, such as that of the logarithm by means of the identity

$$\text{Exp.}\,(\log X) \equiv Y. \qquad \dots \qquad \dots \qquad (18.7)$$

§ 19. *Conditions for functionality.*

In the present paragraph we propose to deduce the conditions that the infinitary variable

$$Y = \sum_{n=1}^{\infty} y_n e_n \qquad \dots \qquad \dots \qquad (19.1)$$

be an analytic function of the independent infinitary variable

$$X = \sum_{n=1}^{\infty} x_n e_n \qquad \dots \qquad \dots \qquad (19.11)$$

We obtain by differentiation

$$\frac{\partial Y}{\partial x_r} = \frac{dY}{dX} \cdot \frac{\partial X}{\partial x_r} = e_r \frac{dY}{dX} = \sum_{n=1}^{\infty} e_n \frac{\partial y_n}{\partial x_r}. \qquad \dots \qquad (19.2)$$

Assume

$$r = kl, \qquad \dots \qquad \dots \qquad (19.21)$$

when (19.2) gives

$$e_l \sum_{n=1}^{\infty} e_n \frac{\partial y_n}{\partial x_{kl}} = e_{kl} \sum_{m=1}^{\infty} e_m \frac{\partial y_m}{\partial x_l}, \qquad \dots \qquad (19.22)$$

or by (17.3)

$$\sum_{n=1}^{\infty} e_n \frac{\partial y_n}{\partial x_{kl}} = \sum_{m=1}^{\infty} e_{km} \frac{\partial y_m}{\partial x_l}. \qquad \dots \qquad \dots \qquad (19{\cdot}221)$$

It follows that the indices of the units in the sum on the left must be multiples of k.

Putting

$$n = kml, \dots \qquad \dots \qquad \dots \qquad (19.222)$$

we obtain the necessary (but not sufficient) condition for functionality

$$\frac{\partial y_{km}}{\partial x_{kl}} = \frac{\partial y_m}{\partial x^l} \dots \qquad \dots \qquad \dots \qquad (19{\cdot}3)$$

In addition, if n is not a multiple of k, (19·221) gives

$$\frac{\partial y_n}{\partial x_{kl}} = 0, \qquad \dots \qquad \dots \qquad \dots \qquad (19{\cdot}31)$$

from which follows, that the derivatives in (19·3) vanish, unless m is a multiple of l, or, by (19·21), (19·222)

$$n = rs \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19{\cdot}4)$$

As the *necessary and sufficient conditions for functionality,* we thus finally obtain, by the aid of (19·3)

$$\frac{\partial y^n}{\partial x_r} = \left\{ \begin{array}{l} = \dfrac{\partial y}{\partial x_1} \\ = 0 \end{array} \right\}, \quad \ldots \quad \ldots \quad (19{\cdot}5)$$

according as (19·4) is, or is not, satisfied.

If these conditions are satisfied, we have

$$\frac{dy}{dx} = \frac{\partial y}{\partial x_1}, \quad \ldots \quad \ldots \quad (19{\cdot}6)$$

by putting $r = 1$ in (19 2).

The relation (19·5), (19·6) may be at once vertified by the aid of (9·3), which may be written

$$f\left\{ \sum_{n=1}^{\infty} x_n \, e_n \right\} = f(x_1)\, e_1 + \frac{x_2}{1!} f'(x_1)\, e_2 + \frac{x_3}{1!} f'(x_1)\, e_3$$

$$+ \left\{ \frac{x_4}{1!} f'(x_1) + \frac{x^2_2}{2!} f''(x_1) \right\} e_4 + \frac{x_5}{1!} f'(x_1)\, e_5$$

$$+ \left\{ \frac{x_6}{1!} f'(x_1) + \frac{x_2 \;\, x_3}{1!\,1!} f''(x_1) \right\} e_6 + \ldots . \quad (19{\cdot}7)$$

§ 20. *Transformation of units.*

Lex X be any given number of our infinitary algebra, which from its connection with Dirichlet's series may be said to be *of Dirichlet's type.*

We may now construct a new infinitary algebra with the units

$$E_n = X^{n-1}, \quad \ldots \quad \ldots \quad (20{\cdot}1)$$

subject to the multiplication table

$$E_m E_n = E_{m+n-1}, \quad \ldots \quad (20{\cdot}2)$$

We may, for example, let

$$X = [\,1\,], \quad \ldots \quad \ldots \quad (203).$$

when

$$\left.\begin{aligned} E_1 &= e_1 = 1 , \\ E_2 &= \sum_{n=1}^{\infty} e_n = \left[\ 1\ \right], \\ E_3 &= \sum_{n=1}^{\infty} t\,(n).\, e_n = \left[\ 1\ \right]^2, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}\right\} \quad \dots \quad \dots \quad (20\ 31)$$

If now the e's are supposed solved with respect to the E's, we may express any number

$$A = \sum_{n=1}^{\infty} a_n\, e_n \quad \dots \quad \dots \quad \dots \quad \dots \quad (20.4)$$

of our initial algebra in the form

$$A = \sum_{n=1}^{\infty} A_n\ E_n \quad \dots \quad \dots \quad \dots \quad \dots \quad (20.41)$$

This process would, however involve a discussion of infinite determinants, which we may avoid by observing that the units E_n may provisionally be defined by means of the equivalence

$$E_n \sim X^{n-1}, \quad \dots \quad \dots \quad \dots \quad (20.11)$$

where X is an *arbitrary* complex number. Just as we, in § 3, established the connection between our initial algebra and the theory of D. S., we may now similarly connect our new algebra with Mac-Laurin's series, and speak of it as *an infinitary algebra of Mac-Laurin's type.*

The operations of addition, subtraction and multiplication of this algebra are defined by

$$A \pm B = \sum_{n=1}^{\infty} (A_n \pm B_n)\ E_n\,, \dots \quad \dots \quad \dots \quad (20.5)$$

$$A\,.\,B = \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} A_i \cdot B_{n-i+1} \right) E_n \quad \dots \quad \dots \quad (20\ 6)$$

The process of division is defined by the formula

$$A^{-1} = \frac{1}{A_1}\Bigg[E_1 - \frac{A_2}{A_1}\, E_2 + \left\{ \left(\frac{A_2}{A_1}\right)^2 - \frac{A_3}{A_1} \right\} E_3 - \left\{ \left(\frac{A_2}{A_1}\right)^3 - 2\,\frac{A_2}{A_1}\cdot\frac{A_3}{A_1} + \frac{A_4}{A_1} \right\} E_4 + \dots \Bigg], \ (20.7)$$

provided the condition

$$A_1 \neq 0 \quad \dots \quad \dots \quad \dots \quad \dots \quad (20.71)$$

is satisfied.

[As in the case of the algebra of Dirichlet's type, certain anomalies with regard to the operation of division appears in the present algebra, which may be partially eliminated by the admission of units with *negative* and *zero* indices, subject to the multiplication table (20.2), an artifice which will, however, introduce complications in other directions.]

The infinitary algebra of Mac-Laurin's type may now be developed on lines similar to those followed in the case of the infinitary algebra of Dirichlet's type. Although the two algebras are equivalent under linear transformation of units in the field considered, the former appears to be of less interest than the latter, and to furnish less elegant results. It will therefore not be considered further here.

THE GROUP-THEORY ELEMENT OF THE HISTORY OF MATHEMATICS*

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FEW mathematical terms suggest such fundamental human cravings as the term group, and few have been more appropriately chosen. Just as human society has led to perplexities which increased with the advance of civilization, so the mathematical group-theory has given rise to problems which became more and more difficult with the advances in the development of mathematics. In both cases the primitive stages are comparatively simple and their history throws important light on the later developments.

The history of the mathematical group-theory can be conveniently divided into three periods. The first of these extends from the beginning of mathematical history to about 1770 A D, and may be called the *implicit period* since the group concept was then employed without being explicitly stated. The second, or *specialization period*, extends from about 1770 to about 1870. During this period the theory of substitution groups was founded as an autonomous science and the usefulness of this theory in the study of algebraic equations was emphasized. The third, or *generalization period*, extends from about 1870 to the present day, and is characterized by increased generalizations by abstraction and the explicit use of groups in each of the large domains of mathematics.

The most fundamental property of the elements of a group in the common restricted sense of this term is that they satisfy the condition that each group contains one and only one element which satisfies the equation

$$xy = z$$

whenever any two of these symbols are replaced by two equal or unequal elements of the group. This condition is evidently satisfied by the members of the number systems of the ancients, including the Babylonians, the Egyptians and the Greeks. On the other hand, the zero of our common modern number system destroys this group property of the entire system. If the zero is excluded the rest of these numbers (real or complex) constitute a group as regards multiplication.

One of the oldest groups of operations in the history of mathematics is the multiplication group whose elements are all the ordinary rational

* From *The Scientific Monthly*, January 1921.

numbers with the exception of zero. This group is used implicitly in the Ahmes papyrus, written about 1700 B. C., since the linear equation and fractions are found frequently in this work. It is an interesting historical fact that for many centuries after this date the ancients seemed to have considered only numbers which are elements of this group, and one is led to inquire to what extent the guiding influence of the group concept was responsible for the late introduction of zero as a number.

It might at first appear that the introduction of zero as a number tended to show that the group concept was not a fundamental guiding principle in the development of arithmetic since it failed to dominate when opposing forces presented themselves. The impression that the group concept did not dominate when zero was introduced is, however, not quite correct, since this number which was rejected by the multiplication group was destined to become the principal element, or the identity, of the addition group. "The stone which the builders rejected, the same is become the head of the corner."

It is an interesting fact that the addition group was made possible by the introduction of zero and by granting full number citizenship to the negative numbers, and the latter was done about the time when the multiplication group was somewhat impaired by the introduction of zero. These important extensions of our number system were completed during the seventeenth century but some steps in this direction had been taken a thousand years earlier especially by the Hindus. In particular, Brahmagupta had already illustrated negative and positive numbers by debts and credits and he observed that a debt subtracted from zero becomes a credit and a credit subtracted from zero becomes a debt, and if one subtracts a debt from a credit or a credit from a debt one obtains the sum.

The long delay in the general introduction of zero as a number seems to show that the ancient peoples held tenaciously to the view that all of the numbers without exception should constitute a group as regards multiplication. It is doubtless true that the group concept was not clearly observed by them and that they could not have given a satisfactory account of the motives which guided them in their efforts for more knowledge about numbers, but the fact that they were guided by this concept seems to be well established.

H. Poincaré pointed out that the same concept guided the ancients in their efforts to secure a knowledge of geometry and he noted that the absence of any direct reference to groups in Euclid's *Elements* was due to the fact that the group notions were among the oldest mathematical notions to be assimilated and hence they did not seem to require explicit mention even at the time of Euclid, notwithstanding the fact that

the principal foundation of Euclid's demonstrations is really the group and its properties.[1] In view of the fact that the group notions are so fundamental and elementary that they did not seem to require explicit mention at the time of Euclid, it may at first appear strange that during the latter half of the nineteenth century these notions assumed a prominent place in the mathematical literature and that the subject of group-theory began to be regarded as one of the most difficult in the whole range of pure mathematics.

The reason for this change of attitude on the part of the mathematicians is not difficult to discover. As long as only the most general notions of groups were needed the subject was naturally regarded as too elementary to require any special attention. The idea that a set of distinct elements should have the property that any two of them can be combined into one and that this one is also found in the set was illustrated not only by the natural numbers but also by the movements of figures in space, and hence this idea became firmly fixed in the human mind at an early age. It is in accord with the human yearnings for completeness and it is a natural extension of the notion of cyclic changes which were illustrated by the daily and the seasonal apparent movements of the sun.

There is only a short step from this idea of completeness to the idea that a set of distinct elements has the property that: when any two of the symbols in the equation $xy=z$ are replaced by distinct or equal elements of the set, the resulting linear equation has always one and only one root in the set. If we add to these conditions the condition that the associative law shall be satisfied when the elements of the set are combined, we have a complete modern definition of the term group, and it is at once apparent that this definition involves only very fundamental and elementary notions in regard to laws in the world of ideas.

While the general laws of the group are very mild, they required exceptions at an early stage in the growth of mathematics. As was noted above, the entrance of zero into our number system required some modification of these laws as regards the operation of multiplication. The one law of combination imposed by the group notion was too narrow for the full development of our operations with numbers, where two modes of combination, now known as addition and multiplication, were developed even in pre-historic times. Historically the group concept may therefore be said to embody fundamental laws of combination with which human beings became acquainted in pre-historic times but which had to be violated in certain respects in order to secure the most fruitful mathematical developments.

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[1] H. Poincaré, "On the Foundations of Geometry," *The Monist*, Vol. 9, (1898) p. 34.

As these laws were not formulated in the abstract until about 1870, their early violators were naturally unconscious of the significance of their steps as regards the group concept. In fact, these laws might never have been formulated if it had not been discovered that without violating any of them it was possible to develop a very useful and extensive body of knowledge. The mathematical world discovered this fact by accident and at a comparatively late date. The discovery seems to have been due to the development of a large body of knowledge relating to a special class of groups now known as *substitution groups.*

The fundamental ideas involved in this body of knowledge are also very elementary. About 1770 J. L. Lagrange and others were much interested in the solution of the general equation in one unknown. More than two centuries had then elapsed since several Italian mathematicians had discovered algebraic solutions of the general cubic and of the general biquadratic equation. All efforts to obtain a solution of the general quintic had failed and the mathematical world was becoming more and more deeply interested in either making further advances along this line, or proving that such advances are impossible.

In the study of the methods which had led to success for the lower degrees it appeared that the number of different formal values which certain unsymmetric rational functions of a number of variables assume when these variables are permuted in every possible manner was of fundamental importance. For instance, the expression $x_1x_2 + x_3x_4$ assumes the following three values

$$x_1x_2+x_3x_4,\ x_1x_3+x_2x_4,\ x_1x_4+x_2x_3$$

when the four variables x_1, x_2, x_3, x_4 are permuted in every possible manner. As there are 24 possible permutations of these four variables eight of them transform such a function into itself. These eight permutations constitute an important substitution group known as the octic group.

In general, all the permutations on n variables which transform into itself a certain rational function of these variables constitute a substitution group. Hence the study of such groups seemed important for the purpose of proving the existence or the non-existence of rational functions of a given number of variables which assume a given number of values when these variables are permuted in every possible manner. The concept of a substitution group on n variables is thus seen to be a very elementary one but the study of such groups led to a large body of theorems. Some of these appeared elegant even if they were supposed to apply only to a rather special field.

[*To be continued.*]

Leaves from a Lecturer's Diary.

The following demonstration for obtaining the recurrence formulæ of the convergents of a continued fraction is short.

If $\frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \dots$ be the continued fraction it is obvious that the nth convergent

$$\frac{p_n}{q_n} = \frac{Aa_n + Bb_n}{Ca_n + Db_n} \quad \dots \quad \dots \quad (\alpha)$$

in lowest terms where A, B, C and D are functions of all the elements up to the $(n - 1)$th.

If in $\frac{p_n}{q_n} = \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \dots \frac{a_{n-2}}{b_{n-2} +} \frac{a_{n-1}}{b_{n-1} +} \frac{a_n}{b_n}$, $a_n = 0$, this reduces to $\frac{p_{n-1}}{q_{n-1}}$, while if $b_n = 0$, $b_{n-1} + \frac{a_n}{b_n}$ becomes infinity and therefore everything after b_{n-2} becomes zero, thus reducing the convergent to $\frac{p_{n-2}}{q_{n-2}}$.

Now on inserting successively $a_n = 0$ and $b_n = 0$ in (α), we get

$$\frac{p_{n-1}}{q_{n-1}} = \frac{B}{D} \text{ and } \frac{p^{n-2}}{q_{n-2}} = \frac{A}{C}.$$

Hence

$$\frac{p_n}{q_n} = \frac{a_n p_{n-2} + b_n p_{n-1}}{a_n q_{n-2} + b_n q_{n-1}}$$

from which the proof may be completed for equating the numerator and denominator separately as in the usual manner.

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It is fairly well-known that the cissoid $y^2(2a - x) = x^3$ can be employed to solve the Delian problem of inserting two mean proportionals between two given numbers. The following analysis of the proof given in Edwards' *Differential Calculus*, § 438, serves to suggest a curve which can be similarly employed to interpolate n mean proportionals. If two means, say α and β, are to be introduced between a and b (say $a > b$), we are required to have $\frac{a}{\alpha} = \frac{\alpha}{\beta} = \frac{\beta}{b}$, *i.e.*, $a^2 b = \alpha^3$ ... ... (1)

Now describe a circle of radius a with A as centre and let OC be a diameter. Upon AD, which is perpendicular to it, choosing AB $= b$ and A$\alpha = \alpha$, we find for the equations of CB and Oα, respectively

$$ay + bx = 2ab \quad \dots \quad \dots \quad (2) \quad \text{and} \quad ay = \alpha x \quad \dots \quad \dots \quad (3)$$

Eliminating α and b, we find for the locus of P, the intersection of (2) and (3), the cissoid $x^3 = y^2(2a - x)$.

Conversely when the cissoid is drawn, we have the construction given by Edwards quoted above. Similarly if n mean proportionals are to be inserted, say α, β ... such that $\frac{a}{\alpha} = \frac{\alpha}{\beta} = \dots \frac{v}{b}$ which gives $\alpha^{n+1} = a^n b$.
Eliminating as above α and b between this and (2) and (3) given above we are led to the curve $\quad x^{n+1} = y^n(2a - x)$.
The converse process of determining α is precisely similar to the above construction when this generalised cissoid is drawn. After α is determined, β, γ, ... are each determined by the ordinary methods from

$$a\beta = \alpha^2;\ \alpha\gamma = \beta^2;\ \dots$$

SOLUTIONS

Question 661.

(S. RAMANUJAN):—Solve in integers $x^3 + y^3 + z^3 = u^6$ and deduce the following :—

(i) $6^3 - 5^3 - 3^3 = 2^6$. (ii) $8^3 + 6^3 + 1^3 = 3^6$.

(iii) $12^3 - 10^3 + 1^3 = 3^6$. (iv) $46^3 - 37^3 - 3^3 = 6^6$.

(v) $174^3 + 133^3 - 45^3 = 14^6$. (vi) $1188^3 - 509^3 - 3^3 = 34^6$.

Solution by Prof. N B. Mitra, M.A.

To solve $x^3 + y^3 + z^3 = u^6$ ... ...(a)

$= v^3$, where $v = u^2$.

(i) A particular solution may be obtained thus :—

We have $x^3 + y^3 = v^3 - z^3$.

Let $x = p + q,\ y = p - q,\ z = r - s,\ v = r + s$.

Then substituting we get, $p(p^2 + 3q^2) = s(s^2 + 3r^2)$.

Put $p = a(ll + 3m),\ q = a(l - m),\ r = bl,\ s = 3bm$,

we get, $4a^3(l + 3m)(l^2 + 3m^2) = 9b^3m(l^2 + 3m^2)$,

so that $4a^3l = 3m(3b^3 - 4a^3)$,

so that we may put $l = 3(3b^3 - 4a^3),\ m = 4a^3$.

Substituting and writing a for $2a$, we get,

$x = a(9b^3 - a^3),\ y = a^4,\ z = 3b(3b^3 - a^3),\ u = 3a^2$.

Putting $a = b$, we get $8^3 + 1^3 + 6^3 = 3^6$ .. (ii).

(ii) Another special solution may be obtained from Q. 441 proposed by the late Mr. Ramanujan (J. I. M. S., Vol. V, page 39).

We have $(3a^2 + 5ab - 5b^2)^3 + (4a^2 - 4ab + 6b^2)^3 + (5a^2 - 5ab - 3b^2)^3$
$= (6a^2 - 4ab + 4b^2)^3$.

A solution of the present problem will be obtained if we can make $6a^2 - 4ab + 4b^2$ a perfect square.

Assume therefore $6a^2 - 4ab + 4b^2 = (ka - 2b)^2$,

whence $a(k^2 - 6) = 4b(k - 1)$.

Hence we can take $a = 4(k - 1),\ b = k^2 - 6$ and we get the following values $x = -5k^4 + 20k^3 + 88k^2 - 216k - 12$

$y = 6k^4 - 16k^3 + 8k^2 - 32k + 184$

$z = -3k^4 - 20k^3 + 136k^2 - 40k - 148$

$u = 2k^2 - 4k + 6$,

Putting

$k=1$, we get $-125^3+150^3-75^3=10^6$ or $-5^3+6^3-3^3=2^6$ ... (i)

$k=2$, $\quad -12^3+120^3+108^3=12^6$ or $1^3-10^3+12^3=3^6$ ... (iii)

$k=0$, $\quad -12^3+184^3-148^3=12^6$ or $-3^3+46^3-37^3=6^6$ ... (iv)

$k=4$, $\quad 532^3+696^3-180^3=28^6$ or $133^3+174^3-45^3=14^6$. (v)

(iii) We may proceed also thus:—Mr. S. Narayanan in his solution of Q. 441 shows (J. I. M. S., Vol. VI, page 226) that if

$$(pa^2+mab-mb^2)^3+(na^2-nab+lb^2)^3+(ma^2-mab-pb^2)^3$$
$$=(la^2-nab+nb^2)^3 \quad \dots \quad (1)$$

then $l=\lambda(\lambda^3+1)$, $m=2\lambda^3-1$, $n=\lambda(\lambda^3-2)$, $p=\lambda^3+1$ ... (2)

Multiply both sides of (1) by l^3;

then the r. h. s. $=(l^2a^2-lnab+lnb^2)^3$.

Assume $l^2a^2-lnab+lnb^2=(la+ka)^2$, whence $a:b=(ln-k^2):(l2k-n)$, so that we may take $a=ln-k^2$, $b=l(2k-n)$ ... (3)

Then a solution of the present problem is

$x=l(pa^2+mab-mb^2)\,q^2$,

$y=l(na^2-nab+lb^2)\,q^2$,

$z=l(ma^2-mab-pb^2)q^2$,

$u=l(ln+k^2-kn)\,q$ where q is any quantity.

To get numerical values, we give any value to λ, then (2) gives us the values of l, m, n, p. These together with any value we may give to k determine a and b by (3). Then we choose q so as to make x, y, z integral. If l is a square, say r^2, then we may omit the factor l from the values of x, y, z and substitute r for l in the value of u. Otherwise the values of x, y, z, u will have a common divisor.

This solution is however still not general.

(iv) We may obtain a more general solution in the following way:—

Euler gives the most general solution of $A^3+B^3+C^3=D^3$ as follows:—

$A=(3ac+3bc-ad+3bd)(d^2+3c^2)-(a^2+3b^2)^2$.

$B=(3ac-3bc+ad+3bd)(d^2+3c^2)+(a^2+3b^2)^2$.

$C=(d^2+3c^2)^2-(3ac+3bc-ad+3bd)(a^2+3b^2)$.

$D=(d^2+3c^2)^2+(3ac-3bc+ad+3bd)(a^2+3b^2)$

so that $x=AD$, $y=BD$, $z=CD$, $u=D$ is a solution of (a).

The defect of this method is that the values of x, y, z, u so obtained have a common divisor.

We may attack the problem in another way yet. Give any values we like to a, b, c, d and thus obtain the values of A, B, C, D. If in any case D is a square, we obtain a solution of (a).

The problem of finding the general solution of

$$(a^2 + 3b^2)(3ac - 3bc + ad + 3bd) + (d^2 + 3c^2)^2 = w^2$$

giving all the sets of values of a, b, c, d, w seems to be intractable.

Question 681.

(S. Ramanujan) :—Solve in integers $x^3 + y^3 + z^3 = 1$ and deduce the following :—

(i) $6^3 + 8^3 = 9^3 - 1$. (ii) $9^3 + 10^3 = 12^3 + 1$.

(iii) $135^3 + 138^3 = 172^3 - 1$. (iv) $791^3 + 812^3 = 1010^3 - 1$.

(v) $11161^3 + 11468^3 = 14258^3 + 1$. (vi) $65601^3 + 67402^3 = 83802^3 + 1$.

Solution by N. B. Mitra.

A particular solution may be obtained in the following way :—

Put $x = p + q$, $y = p - q$, $z = 2r - 1$.

Then $p(p^2 + 3q^2) = (1 - r)(4r^2 - 2r + 1)$.

Let $p = 3a$, $q = a(4r - 1)$. (This introduces a restriction).

Then we get after reduction and cancellation of the factor $4r^2 - 2r + 1$,

$$r = 1 - 36a^3,\ q = 3a(1 - 48a^3)$$

whence

$$x = 6a(1 - 24a^3),\ y = 144a^4,$$

$$z = 1 - 72a^3$$

or writing a for $2a$,

$$x = 3a(1 - 3a^3),$$

$$y = 9a^4,$$

$$z = 1 - 9a^3.$$

Putting $a \pm 1$, we get $-6^3 + 9^3 - 8^3 = 1$ ... ... (i)

$a = -1$, $-12^3 + 9^3 + 10^3 = 1$ ... ... (ii)

Question 1074.

(V. Thiruvenkatachari) :—If $a_1, a_2, a_3, \ldots a_n$ be n unequal positive numbers, show that

(i) $L_n = (G_1 G_3 G_5 \ldots)/(G_2 G_4 \ldots)$,

(ii) $G_n = (L_1 L_3 L_5 \ldots)/(L_2 L_4 \ldots)$,

where G_r denotes the product of the G.C.M.'s of the numbers r at a time and L_r denotes the similar product of the L.C.M.'s.

Solution by P. R. Venkatakrishna Iyer.

Let $a_1 = \dfrac{\rho_1 PG}{g_1}$, $a_2 = \dfrac{\rho_2 PG}{g_2}$, ... $a_r = \dfrac{\rho_r PG}{g_r}$, .. ,

where G is the G.C.M. of $a_1, a_2, \ldots a_n$, and $P = g_1 g_2 \ldots g_n$, (one or more of the numbers $g_1, g_2\, g_n \ldots$ may be equal to unity). Also $\rho_1, \rho_2, \rho_3 \ldots \rho_n$ and $g_1, g_2, g_3 \ldots g_n$ are all prime to each other.

Denoting the product $\rho_1 \rho_2 \ldots \rho_n$ by the letter p,

we have $\qquad G_1 = p \,.\, P^{n-1} G^n,$

$$G_2 = \frac{P^{{}^nC_2}\, G^{{}^nC_2}}{P^{n-1}} = P^{{}^{n-1}C_2}\, G^{{}^nC_2},$$

similarly $\qquad G_3 = P^{{}^{n-1}C_3}\, G^{{}^nC_3}$

... ...

$$G_r = \frac{P^{{}^nC_r}\, G^{{}^nC_r}}{P^{{}^{n-1}C_{r-1}}} = P^{{}^{n-1}C_r} \,.\, G^{{}^nC_r}$$

$$G_{n-2} = P^{{}^{n-1}C_1}\, G^{{}^nC_2}$$

$$G_{n-1} = P \,.\, G^{{}^nC_1}$$

$$G_n = G.$$

When n is even, $\dfrac{G_1 \,.\, G_3 \,.\, G_5 \ldots G_{n-1}}{G_2 \,.\, G_4 \ldots \ldots G_n}$

$$= p \,.\, \frac{P^{{}^{n-1}C_1 + {}^{n-1}C_3 + \ldots + {}^{n-1}C_0}}{P^{{}^{n-1}C_2 + {}^{n-1}C_4 + \ldots + {}^{n-1}C_1}} \times \frac{G^{{}^nC_1 + {}^nC_3 + \ldots + {}^nC_1}}{G^{{}^nC_2 + {}^nC_4 + \ldots + {}^nC_0}}$$

$= p \,.\, \dfrac{P^{2^{n-2}}}{P^{2^{n-2}-1}} \times \dfrac{G^{2^{n-1}}}{G^{2^{n-1}-1}} = p \,.\, P \,.\, G = L_n$, L_n being the L.C.M. of $a_1, a_2, \ldots a_n$.

Again, $\qquad L_1 = p \,.\, P^{n-1} G^n$

$$L_2 = p^{n-1} \,.\, (PG)^{{}^nC_2}$$

$$L_3 = p^{{}^{n-1}C_1} \,.\, (PG)^{{}^nC_3}$$

... ...

$$L_r = p \,.^{\,{}^{u-1}C_{r-1}} \,.\, (PG)^{{}^nC_r}$$

... ...

$$L_{n-2} = p^{{}^{n-2}C_2} (PG)^{{}^{n}C_2}$$
$$L_{n-1} = p^{n-1} (PG)^{{}^{n}C_1}$$
$$L_n = p \cdot P \cdot G.$$

When n is even, $\dfrac{L_1 \cdot L_3 \cdot L_5 \ldots L_{n-1}}{L_1 \cdot L_4 \ldots \ldots L_n}$

$$= \frac{p^{{}^{n-1}C_0 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_1}}{p^{{}^{n-1}C_1 + {}^{n-1}C_3 + \ldots + {}^{n-1}C_2 + {}^{n-1}C_0}} \times \frac{p^{n-1 + {}^{n}C_3 + {}^{n}C_5 + \ldots + {}^{n}C_1}}{p^{{}^{n}C_2 + {}^{n}C_4 + \ldots + {}^{n}C_2 + {}^{n}C_0}}$$
$$\times \frac{G^{{}^{n}C_1 + {}^{n}C_3 + \ldots + {}^{n}C_1}}{G^{{}^{n}C_2 + {}^{n}C_4 + \ldots + {}^{n}C_2 + {}^{n}C_0}}$$

$$= \frac{p^{2^{n-2}}}{p^{2^{n-2}}} \times \frac{P^{2^{n-1}-1}}{P^{2^{n-1}-1}} \times \frac{G^{2^{n-1}}}{G^{2^{n-1}-1}} = G = G_n.$$

The results are true when n is odd.

Question 1081.

(K. J. SANJANA, M.A.) :—If a, b, c be real unequal quantities whose sum is zero, prove that the equation

$$4\,(a^2 + b^2 + c^2)\,x^3 + 18\,abc\,x^2 - abc\,(a^2 + b^2 + c^2) = 0$$

has two imaginary or complex roots.

Solution by Hemraj.

The Hessian of the cubic is

$$18\,abc\,x^2 + 2\,(a^2 + b^2 + c^2)^2\,x + 3\,abc\,(a^2 + b^2 + c^2) = 3$$

The roots of the Hessian are real if

$$(a^2 + b^2 + c^2)^3 > 54\,a^2b^2c^2$$

since $\quad a^2 + b^2 + c^2 > 0$

i.e. if $\quad 27a^2b^2c^2 + 4\,(bc - a^2)^3 < 0$

i.e. if $\quad 15a^2b^2c^3 + 4\,(b^3c^3 + 3a^4bc - a^6) < 0$

i.e. if $\quad 3a^2b^2\,(a^2 + b^2) - 12ab\,(a^4 + b^4)$
$$- 4\,(a^6 + b^6) + 26\,a^3\,b^3 < 0$$

i.e. if $\quad 3a^2b^2 - 12ab\,(a + b)^2 - 4\,(a^2 + ab + b^2)^2 < 0$

since $\quad (a - b)^2 < 0$

i.e. if $\quad 4\,(a + b)^4 - 4\,ab\,(a + b)^2 + a^2b^2 > 0$

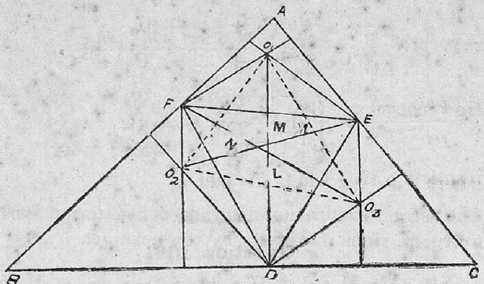
i.e. if $\quad (2c^2 - ab)^2 > 0$ which is so.

Hence the cubic has two imaginary or complex roots.

Question 1087.

(A. A. KRISHNASWAMI AIYANGAR) :—ABC is any triangle. DEF is a triangle inscribed within it, D, E, F lying on the sides BC, CA, AB respectively, and O_1, O_2, O_3 are the orthocentres of the triangles AEF, BDF, CED. Prove that (i) the area of the triangle $O_1O_2O_3$ is equal to that of the triangle DEF and (ii) the two triangles $O_1O_2O_3$ and DEF can be inscribed in the same conic.

Solution by Hemraj and Tiruvenkatachari.



Let O_1, O_2, O_3, be the orthocentres.
Let O_1 D, O_2 E, O_3 F form a triangle LMN.
Since $O_1F \parallel O_3D$, $FO_2 \parallel EO_3$, $O_1E \parallel DO_2$.

(i) $\therefore$ $\Delta\, O_1FO_3 = \Delta\, O_1FD$
i.e. $\Delta\, O_1LO_3 = \Delta\, FLD$
and $\Delta\, O_1MO_2 = \Delta\, EMD$
$\Delta\, O_2NO_3 = \Delta\, ENF$
Δ LMN $= \Delta$ LMN

Adding we get $\Delta\, O_1O_2O_3 = DEF$

(ii) $O_1FO_2DO_3E$ is a hexagon whose opposite sides are parallel,
$\therefore$ the points of intersections of the opposite sides are collinear.
Hence O_1, F, O_2, D, O_3, E lie on a conic.

Question 1089.

(S. MAHADEVAN) :—Prove that if $|x| < 1$

$$\frac{x}{(1+x)^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^3}{(1+x^3)^2} + \ldots\ldots$$

$$= \frac{x}{1-x} - \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} - \ldots\ldots = \sum_{1}^{\infty} \psi(n)\, x^n$$

where $\psi(n)$ denotes the excess of the odd over the even divisors of n, 1 and n being reckoned as divisors.

Solution by Hemraj.

Expand the terms of the 1st series and place them upon a summation diagram.

$\frac{x}{(1+x)^2}$	x	$-2x^2$	$3x^3$	$-4x^4$	$5x^5$	. . .
$\frac{x^2}{(1+x^2)^2}$	x^2	$-2x^4$	$3x^6$	$-4x^8$	$5x^{10}$	. . .
$\frac{x^3}{(1+x^3)^2}$	x^3	$-2x^6$	$3x^9$	$-4x^{12}$	.	. . .
. .	.	.	.	.	.	. . .

Now consider

X	$2X^2$	$3X^3$	$4X^4$	. .
X^2	$2X^4$	$3X^6$	$4X^8$	. .
X^3	$2X^6$	$3X^9$	$4X^{12}$	. .

where $X = |x|$

The second summation-diagram, consisting of positive numbers only is convergent, since if we sum by rows, we get

$$\frac{X}{(1-X)^2} + \frac{X^2}{(1-X^2)^2} + \frac{X}{(1-X^3)^2} + \ldots\ldots$$

the limit of whose convergency-ratio is

$$\underset{n \to \infty}{Lt.} \frac{X^{n+1}}{(1-X^{n+1})^2} \times \frac{(1-X^n)^2}{X} = X < 1.$$

Hence the original summation diagram is absolutely convergent and may therefore be summed by any convenient system of curves.

If we sum by rows, we obtain $\frac{x}{(1+x)^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^3}{(1+x^3)^2} +$ (1)

If we sum by columns, we get $\frac{x}{1-x} - \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} - \ldots\ldots$ (2)

Hence (1) and (2) are equal.

If the above double series be arranged according to powers of x, we sum it by columns. Since the double series still converges after every term is made positive

$\therefore$ the sums by rows and columns are certainly equal;

$$\therefore \ (1) = x - x^2 + 4x^3 - 5x^4 + \ldots\ldots = \sum_{1}^{\infty} \psi(n)\, x^n.$$

Question 1090.

(S. MAHADEVAN) :—The focus S is joined to any point P_1 on the ellipse, P_1S cuts the ellipse again in Q_1, Q_1S' cuts the ellipse again in P_2, and so on. Show that SP_n ultimately tends to coincide with the major axis.

Solution by Hemraj and Kewalramani.

Let the eccentric angles of $P_1, Q_1, P_2, Q_2 \ldots\ldots, Q_{n-1}, P_n$ be $\theta_1, \phi_1, \theta_2, \phi_2 \ldots\ldots \phi_{n-1}, \theta_n$. Then it is easy to show that

$$\tan\frac{\theta_1}{2}\tan\frac{\phi_1}{2} = \cot\frac{\phi_1}{2}\cot\frac{\theta_2}{2} = \tan\frac{\theta_2}{2}\tan\frac{\phi_2}{2} = \ldots\ldots$$

$$= \cot\frac{\phi_{n-1}}{2}\cot\frac{\theta_n}{2} = \lambda \text{ (say).}$$

$$\therefore \quad \tan\frac{\theta_1}{2}\cot\frac{\theta_n}{2} = \lambda^{2(n-1)}$$

Now $\tan\frac{\theta_1}{2}\tan\frac{\phi_1}{2} = \lambda \ (\neq 1)$.

$\therefore \quad \theta_n \to 0$ or π according as $\lambda \gtrless 1$,

$\therefore \quad SP_n$ tends to coincide with the major axis.

Question 1091.

(ENQUIRER) :—If a, b, c be the radii of three circles which touch one another externally, and r_1 and r_2 be the radii of the two circles, that can be drawn to touch these three, prove that

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c}.$$

Solution (1) *by Hemraj and Martyn M. Thomas.*

(2) *by N. Sundaram Aiyar.*

(1) Let A, B, C be the centres of the circles a, b, c. Take A as the origin and AB as the x — axis. Then the equations of the circles are

$$x^2 + y^2 - a^2 = 0$$

$$x^2 + y^2 - 2(a+b)x + a^2 + 2ab = 0$$

$$x^2 + y^2 + 2gx + 2fy + c' = 0$$

where $c' = a^2 + 2ac$,

$$g = -\left(a + \frac{a-b}{a+b}c\right)$$

and $\quad f^2 = \dfrac{4abc\,(a+b+c)}{(a+b)^3}$ ... ... (1)

If they are touched by a fourth circle excluding them all, then let its equation be

$$x^2 + y^2 + 2Gx + 2Fy + C = 0$$

where $C = a^2 + 2ra$

$$G = -\left(a + \frac{a-b}{a+b}\,r\right),$$

$$F^2 = \left\{ab\,(r+c) - rc\,\frac{(a^2+b^2)}{a+b}\right\}^2 \times \frac{1}{abc\,(a+b+c)}.$$

[These follow from the condition $c_1 + c_2 - 2r_1r_2 - 2g_1g_2 - 2f_1f_2 = 0$].

Also from (1) $F^2 = \dfrac{4abr\,(r+a+b)}{(a+b)^2}$.

Hence equating the values of F^2 we get

$$r^2\left\{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) - \left(\frac{2}{bc}+\frac{2}{ca}+\frac{2}{ab}\right)\right\}$$
$$- 2r\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + 1 = 0.$$

But if the *first* three circles are touched by a circle enclosing them all, its radius is obtained by changing r into $-r$.

$$\therefore \frac{1}{r_1} = 2\sqrt{\Sigma\frac{1}{bc}} + \Sigma\frac{1}{a} \text{ and } \frac{1}{r_2} = 2\sqrt{\Sigma\frac{1}{bc}} - \Sigma\frac{1}{a}$$

$$\therefore \frac{1}{r_1} - \frac{1}{r_2} = 2\,\Sigma\,\frac{1}{a}.$$

(2) Let A, B, C be respectively the centres of circles whose radii are a, b and c. Let O be the centre of either of the circles which touch the given circles.

Call the angles BOC, COA, AOB in order θ_1, θ_2 and θ_3 respectively,

Then $\theta_1 + \theta_2 + \theta_3 = 2\pi$ ... ... (1)

Let OA, OB, and OC be equal to α, β and γ respectively

If r_2 be the radius of the circle which touches the given $\odot^s$ internally, and r_1 that of the other circle, we have

$r_2 = a + \alpha = b + \beta = c + \gamma$ ... ... (2)

and $\quad r_1 = \alpha - a = \beta - b = \gamma - c$ ... ... (3)

In both cases $(a-b)^2 = (\alpha - \beta)^2$ and two similar relations hold good.

Now $AB^2 = (a+b)^2 = \alpha^2 + \beta^2 - 2\alpha\beta \cos\theta_3$

$$= (\alpha - \beta)^2 + 4\alpha\beta \sin^2 \frac{\theta_3}{2}$$

$$\therefore \quad ab = \alpha\beta \sin^2 \frac{\theta_3}{2}.$$

From this and two similar relations we easily get

$$c = \gamma \frac{\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}}{\sin \frac{\theta_3}{2}}$$

or from (1), $$\frac{\gamma}{c} = \frac{\sin \frac{\theta_1 + \theta_2}{2}}{\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}} = \cot \frac{\theta_1}{2} + \cot \frac{\theta_2}{2} \quad \dots \quad \dots \quad (4)$$

From (4) and two similar results, we have

$$\left(\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c}\right)^2 = 4\left(\Sigma \cot \frac{\theta_1}{2}\right)^2$$

$$= 4 \Sigma \cot^2 \frac{\theta_1}{2} + 8 \Sigma \cot \frac{\theta_2}{2} \cot \frac{\theta_3}{2}$$

$$= 2 \Sigma \left(\cot \frac{\theta_1}{2} + \cot \frac{\theta_2}{2}\right)^2 + 4 \Sigma \cot \frac{\theta_2}{2} \cot \frac{\theta_3}{2}$$

$$= 2\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}\right) + 4 \quad \text{from (4) and (1).}$$

From this relation, as well as (2) and (3), if r stands for r_1 or r_2, we have $\frac{\alpha}{a} = \frac{r}{a} \pm 1$ and two similar relations.

$\therefore$ We have, since $$\Sigma \frac{\alpha^2}{a^2} - 2\Sigma \frac{\beta\gamma}{bc} + 4 = 0,$$

$$\Sigma \frac{r^2}{a^2} \pm 2\Sigma \frac{r}{a} + 3 - 2\left\{\Sigma \frac{r^2}{bc} \pm 2\Sigma \frac{r}{a} + 3\right\} + 4 = 0$$

or $$r^2\left(\Sigma \frac{1}{a^2} - 2\Sigma \frac{1}{bc}\right) \mp 2r \Sigma \frac{1}{a} + 1 = 0$$

Hence r_1 takes the negative and r_2 the positive sign

$$\therefore \frac{1}{r_1^2} - \frac{2}{r_1} \Sigma \frac{1}{a} = -\left(\Sigma \frac{1}{a^2} - 2\Sigma \frac{1}{bc}\right) = \frac{1}{r_2^2} + \frac{2}{r_2} \Sigma \frac{1}{a}$$

$$\therefore \frac{1}{r_1^2} - \frac{1}{r_2^2} = 2\left(\Sigma \frac{1}{a}\right)\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

$$\therefore \frac{1}{r_1} - \frac{1}{r_2} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c}.$$

Question 1092.

(ENQUIRER) :—Find the relation existing between α, β, γ, δ, if

$$\int \frac{\alpha + \beta x + \gamma x^2 + \delta x^3}{\sqrt{x(1-x)}}\, dx$$

does not depend on circular functions.

Solution by Hemraj and S. R. Ranganathan.

$$\int \frac{dx}{\sqrt{x - x^2}} = \cos^{-1}(1 - 2x) = \text{vers}^{-1}\, 2x$$

$$\int \frac{\alpha + \beta x + \gamma x^2 + \delta x^3}{\sqrt{x - x^2}}\, dx = \alpha \int \frac{dx}{\sqrt{x - x^2}} + \frac{\beta}{2} \int \frac{1 - (1 - 2x)}{\sqrt{x - x^2}}\, dx$$

$$+ \gamma \int \frac{x - x(1-x)}{\sqrt{x(1-x)}}\, dx + \delta \int \frac{x - x(1-x) - x^2(1-x)}{\sqrt{x(1-x)}}\, dx$$

$$= \alpha\, \text{vers}^{-1}\, 2x + \frac{\beta}{2}\, \text{vers}^{-1} . 2x - \beta \sqrt{x - x^2} + \frac{\gamma}{2}\, \text{vers}^{-1}\, 2x - \gamma \sqrt{x - x^2}$$

$$+ \gamma \left\{ \frac{(1-2x)\sqrt{x - x^2}}{4} + \tfrac{1}{8} \sin^{-1}(1 - 2x) \right\} + \frac{\delta}{2}\, \text{vers}^{-1}\, 2x - \delta \sqrt{x - x^2}$$

$$+ \frac{3\delta}{2} \left\{ \frac{(1 - 2x)\sqrt{x - x^2}}{4} + \tfrac{1}{8} \sin^{-1}(1 - 2x) \right\} + \frac{\delta}{2} (x - x^2)^{\frac{3}{2}}$$

since $\sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}$.

∴ if this integral is independent of circular functions, then

$$\alpha + \frac{\beta}{2} + \frac{\gamma}{2} - \frac{\gamma}{8} + \frac{\delta}{2} - \frac{3}{16}\delta = 0,$$

i.e.
$$16\alpha + 8\beta + 6\gamma + 5\delta = 0.$$

Question 1093.

(MARTYN M. THOMAS, M.A.) :—A fixed ray of light falls on a plane mirror revolving about an axis in its plane. Show that the locus, in the plane, of the point of incidence is a conic, and that the reflected rays generate a ruled conicoid.

Solution by N. Sundaram, N. B. Mitra and N. B. Pendse.

Let AO be the axis, OP the shortest line between the axis and the fixed ray, and PQ the fixed ray, inclined to the plane AOP at an angle λ, say :

In the fist place, let OA be taken as the y-axis, and the straight line through O, in the revolving plane perpendicular to OA as the x-axis. Let the revolving plane cut the ray at Q in a certain position. Let QR be

perpendicular to plane AOP, and RS to OA. Then the co-ordinates of Q, the point of incidence are (SQ, OS).

$$\text{Now } SQ^2 = SR^2 + RQ^2 \text{ and } \frac{RQ}{RP} = \tan\lambda.$$

∴ the locus of Q in the mirror is

$$x^2 = \alpha^2 + y^2 \tan^2\lambda, \text{ where } \alpha \text{ is SR or OP,}$$

i.e. $$x^2 - y^2\tan^2\lambda = \alpha^2 \text{ which is a conic.}$$

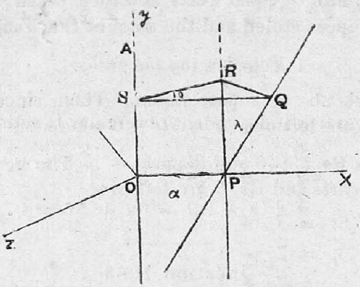
To find the locus of the reflected rays, let OP be the x-axis, OA the y-axis, and the line through O perpendicular to plane AOP, the z-axis Q being the point of incidence, if QSR $= \theta$, the equation to the revolving plane is $$z = x\tan\theta.$$

The equations to PQ are $x = \alpha$, $z = y\tan\lambda$.

∴ the co-ordinates of Q are $(\alpha, \alpha\tan\theta\cot\lambda, \alpha\tan\theta)$.

The co-ordinates of the image of P are obviously,

$$(\alpha\cos 2\theta, 0, \alpha\sin 2\theta).$$



The equations to the reflected ray which is the join of these two points are therefore

$$\frac{x-\alpha}{2\sin^2\theta} = \frac{y - \alpha\tan\theta\cot\lambda}{\tan\theta\cot\lambda} = \frac{z - \alpha\tan\theta}{\left(\tan\theta - \dfrac{2\tan\theta}{1+\tan^2\theta}\right)}$$

or $$\frac{x-\alpha}{\dfrac{2t^2}{1+t^2}} = \frac{y\tan\lambda - \alpha t}{t} = \frac{z-\alpha t}{\dfrac{t(t^2-1)}{t^2+1}} \text{ where } t = \tan\theta$$

or $$\frac{x-\alpha}{2t} = \frac{y\tan\lambda - \alpha t}{1+t^2} = \frac{z-\alpha t}{t^2-1}.$$

Each is equal to $\dfrac{y \tan \lambda - z}{2}$ or $\dfrac{y \tan \lambda + z}{2t^2} - \dfrac{2a}{2t}$.

$$\therefore \quad \frac{x - a}{t} = y \tan \lambda - z$$

and $$(x + a)\, t = y \tan \lambda + z.$$

$\therefore$ Locus of the reflected rays is the ruled conicoid

$$x^2 - a^2 = y^2 \tan^2 \lambda - z^2$$

Question 1095.

(R. S. NARASIMHAM) :—Seven thieves A, B, C, ... secure a sum of rupees, but are obliged to conceal it without counting it. A returns alone, divides the sum into 7 parts, finds that there are 6 rupeees over, takes the 6 rupees and one-seventh part and departs. B returns alone and does the same with the diminished sum. He divides it into 7 parts, finds that there are 5 rupees over, takes the 5 rupees and one-seventh part and departs. Each does the same, the successive remainders being 6, 5, 4, 3, 2, 1, 0. Finally, they all come together and divide the remainder which is a multiple of 7, into 7 equal parts and each takes one part. Find the least number of rupees stolen and the amount that each gets.

Solution (by the author).

Let x be the no. of rupees stolen. Then since it is divided into 7 parts and Rs. 6 are left over $x = 7t + 6$ where t is a +ve integer.

Now A takes Rs. $t + 6$ and departs. $\therefore$ The remainder is $6t$ which is divided into 7 parts and Rs. 5 are left over.

$$\therefore\ 6t = 7p + 5.$$

Similarly
$$\begin{aligned} 6p &= 7q + 4 \\ 6q &= 7r + 3 \\ 6r &= 7s + 2 \\ 6s &= 7u + 1 \\ 6u &= 7v \\ 6v &= 7y, \end{aligned}$$
where $t, p, q \ldots\ldots y$ are +ve integers.

By eliminating $t, p, \ldots v$, we get $279936x - 5764801y = 7812540$.

By writing down the successive convergents of $\dfrac{5764801}{279936}$, we find

$$279936x - 5764801y = 7812540(123073 \times 5764801 - 2534477 \times 279936).$$

so that

$$279936\,(x + 7812540 \times 2534477) = 5764801\,(y + 7812540 \times 123073),$$

i.e. $$\frac{x + 7812540 \times 2534477}{5764801} = \frac{y + 7812540 \times 123073}{279936} = k \text{ (say)}.$$

$$\therefore \left.\begin{aligned} x &= 5764801\, k - (\,)\,(\,) \\ y &= 279936\, k - (\,)\,(\,) \end{aligned}\right\}.$$

Solving these two equations we find that the least positive integral value of k in order that both x and y may be positive integers is 3434780.

The corresponding value of x, *i.e.*, the least no. of rupees that must have been stolen = **4941180**.

The amount that each got is :

$$\left.\begin{aligned} A = 945828;\ B = 844986;\ C = 758550;\ D = 684462; \\ E = 620958;\ F = 566526;\ G = 519870. \end{aligned}\right\} \textit{Ans.}$$

Similar solutions from N. B. Mitra and several others.

Question 1096.

(T. P. Trivedi, M.A., LL.B.) :—Show that if p is prime and n any positive integer, $(np\,!) - n\,!\,(p\,!)^n$ is a multiple of p^{n+3}.

Solution by N. B. Mitra and N. B. Pendse.

Let the notation $M(p)$ denote a multiple of p.

Now $np\,!$ can be written in the form

$$\begin{array}{llllll} 1. & 2. & 3. & \ldots & \ldots & (p-1)\,.\,p. \\ (p+1). & (p+2). & (p+3). & & \ldots & (p+p-1)\,.\,2p. \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \{(n-1)\,p+1\}\,. & \{(n-1)\,p+2\} & \ldots & & \{(n-1)(p+p-1\} & .\,np. \end{array}$$

Now the product of the factors in the last column is $n\,!\,p^n$.

Hence $np\,! = n\,!\,p^n \prod\limits_{r=0}^{r=n-1} f(rp)$,

where $f(rp) = (rp+1)\,(rp+2) \quad \ldots \quad \ldots \quad (rp+p-1)$.

Now, $f(rp) = (rp)^{p-1} + A_1\,(rp)^{p-2} + A_2\,(rp)^{p-3} + \ldots$
$\qquad + A_{p-2}\,rp + A_{p-1} \quad \ldots \quad (1),$

where it is known that all the odd A's except A_1 are $M\,(p^2)$ and all the remaining A's except A_{p-1} are $M\,(p)$ and $A_{p-1} = (p-1)\,!$ Hence, if $p > 3$, $p-2$ is odd since p is prime. Therefore $A_{p-2} = M\,(p^2)$.

$\therefore\ f(rp) = M\,(p^3) + (p-1)\,!$

Hence $\overset{n-1}{\Pi} f(rp) = M(p^3) + \{(p-1)!\}^n$.

Therefore from (1), $(np)! = n!\, p^n \{M(p^3) + (p-1)!\}^n$

$$= M(p^{n+3}) + n!\,(p!)^n.$$

$$\therefore \quad (np)! - n!\,(p!)^n = M(p^{n+3}).$$

Question 1097.

(T. P. Trivedi, M.A., LL.B.) :— Prove that

$$\text{(i)} \quad \left[\int_0^\infty e^{-x^4}\,dx\right]^2 = \int_0^\infty e^{-x^2}\,dx \times \int_0^{\pi/2} \frac{d\theta}{(4-2\sin^2\theta)^{\frac{1}{2}}};$$

$$\text{(ii)} \quad \left[\int_0^\infty e^{-x^6}\,dx\right]^2 = \int_0^\infty e^{-x^2}\,dx \times \int_0^{\pi/2} \frac{d\theta}{(8-6\sin^2\theta)^{\frac{1}{2}}}.$$

Solution by K. J. Sanjana, M.A.

Putting $x^{2n} = y$, it is readily seen that

$$\int_0^\infty e^{-x^{2n}}\,dx \text{ or } I_{2n} \text{ is } \frac{1}{2n}\,\Gamma\left(\frac{1}{2n}\right).$$

Hence

$$\frac{I^2_{2n}}{I_n} = \frac{1}{4n}\,\frac{\left\{\Gamma\left(\frac{1}{2n}\right)\right\}^2}{\Gamma\left(\frac{1}{n}\right)}.$$

When $n = 2$, we get

$$\frac{I^2_4}{I_2} = \frac{1}{8}\,\frac{\{\Gamma(\frac{1}{4})\}^2}{\sqrt{\pi}} = \frac{1}{8\sqrt{\pi}}\,.\,4\sqrt{\pi}\,K\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{d\theta}{(1-\frac{1}{2}\sin^2\theta)^{\frac{1}{2}}}.$$

This gives (i); see *Roberts*, § 134.

Similarly, we have

$$\frac{I^2_6}{I_3} = \frac{1}{12}\,\frac{\{\Gamma(\frac{1}{6})\}^2}{\Gamma(\frac{1}{3})} = \frac{1}{12}\,.\,3\sqrt{2}\,K\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{d\theta}{(1-\frac{3}{4}\sin^2\theta)^{\frac{1}{2}}}, \text{ which gives (ii).}$$

The latter result may be obtained as in Ex. 56, Ch. VI of *Roberts*.

The result for $I^2_8 \div I_4$ could be obtained without much difficulty.

Question 1098.

(K. J. SANJANA, M.A.):—(i) If the circle of curvature at a point of the parabola $y^2 = 4ax$ touches the directrix, the point lies at the intersection of the parabola with the straight line $4x - 5a = 0$.

(ii) If the circle of curvature at a point of the hyperbola $4xy = a^2 + b^2$ touches one of the asymptotes, the point lies at the intersection of the hyperbola with one of the curves

$$3x^4 - 2(a^2 - b^2)\,x^2 + 6x^2y^2 - y^4 = 0.$$
$$3y^4 - 2(a^2 - b^2)\,y^2 + 6y^2x^2 - x^4 = 0.$$

Solution by Martyn M. Thomas, C. N. Sreenivasa Iyengar, S. V. Venkatroya Sastri, O. Ranganathan, V. Ramachandran, N. Sundaram and N. B. Mitra.

First Part.—At the point $(at^2, 2at)$ on the parabola, the x-co-ordinate of the centre of curvature is $2a + 3at^2$; and $\rho = 2a(1 + t^2)^{3/2}$,

$\therefore$ If the directrix is to touch the circle of curvature,

$$(2a + 3at^2) + a = \rho = 2a\,(1 + t^2)^{3/2}$$

$\therefore\;\; 3(1 + t^2) = 2(1+t^2)^{3/2} \quad \therefore\;\; (1 + t^2)^{\frac{1}{2}} = \frac{3}{2}. \quad \therefore\;\; t^2 = \frac{5}{4}.$

$\therefore$ The point $(at^2, 2at)$ lies on the line $x = \dfrac{5a}{4}$.

Second Part.—Let $c^2 = \dfrac{a^2 + b^2}{4}$, so that $\left(ct, \dfrac{c}{t}\right)$ is a point on the hyperbola $xy = c^2$, the equation being referred to its asymptotes as axes of oblique co-ordinates.

If ω be the angle between the asymptotes

$$\cos\omega = \frac{1 - \tan^2\frac{\omega}{2}}{1 + \tan^2\frac{\omega}{2}} = \frac{1 - \dfrac{b^2}{a^2}}{1 + \dfrac{b^2}{a^2}} = \frac{a^2 - b^2}{a^2 + b^2}.$$

Normal at $\left(ct, \dfrac{c}{t}\right)$ is $y(1 - t^2\cos\omega) - x(t^2 - \cos\omega) = \dfrac{c}{t}(1 - t^4)$.

Normal at $\left(ct_1, \dfrac{c}{t_1}\right)$ is $y(1 - t^2_1\cos\omega) - x(t^2_1 - \cos\omega) = \dfrac{c}{t_1}(1 - t^4_1)$.

Eliminating x,

$$y(t + t_1)\sin^2\omega = \frac{c}{tt_1}\{\, t^3 + tt_1 + t^3_1 - \cos\omega + t^3t^3_1 - \cos\omega\, tt_1(t^2_1 + tt_1 + t^2) \,\}.$$

Putting $t_1 = t$, the y co-ordinate of the centre of curvature at $\left(ct, \frac{c}{t} \right)$ is given by $\bar{y} \cdot 2t \sin^2 \omega = \frac{c}{t^3} (3t^2 - \cos \omega + t^6 - 3t^4 \cos \omega)$.

Also $\rho = \frac{(CD)^3}{ab} = \frac{(CP^2 - a^2 + b^2)^{3/2}}{ab}$

$$= \left\{ c^2t^2 + \frac{c^2}{t^2} - 2(ct)\left(\frac{c}{t}\right) \cos (\pi - \omega) - a^2 + b^2 \right\}^{3/2} \Big/ ab$$

$$= \left\{ c^2(t^4 + 1) + t^2(2c^2 \cos \omega - a^2 + b^2) \right\}^{3/2} / abt^3$$

$$= \left\{ (t^4 + 1)(a^2 + b^2) - 2t^2(a^2 - b^2) \right\}^{3/2} / 8abt^3.$$

If the circle of curvature touches the x-axis, we have $\bar{y} \sin \omega = \rho$.

$$\therefore \frac{c\{3t^2 + t^6 - \cos \omega (1 + 3t^4)\}}{2t^3 \sin \omega} = \frac{\{(t^4 + 1)(a^2 + b^2) - 2t^2(a^2 - b^2)\}^{3/2}}{8abt^3}.$$

Since $\sin \omega = \frac{2ab}{a^2 + b^2}$ and $\cos \omega = \frac{a^2 - b^2}{a^2 + b^2}$,

$$(a^2 + b^2) \{ (3t^2 + t^6)(a^2 + b^2) - (1 + 3t^4)(a^2 - b^2) \}^2 = \{ (t^4 + 1)(a^2 + b^2) - 2t^2(a^2 - b^2) \}^3.$$

Simplifying, and removing the common factor $4a^2b^2$,

$$t^8 \cdot 3(a^2+b^2) - 8t^6(a^2 - b^2) + 6t^4(a^2+b^2) - (a^2+b^2) = 0.$$

$\therefore$ Dividing by t^4 and putting $a^2 + b^2 = 4c^2$,

$$3t^4 \cdot 4c^2 - 8t^2(a^2 - b^2) + 6 \cdot 4c^2 - \frac{4c^2}{t^4} = 0.$$

Multiplying by $\frac{c^2}{4}$ and putting x for ct and y for $\frac{c}{t}$,

$$3x^4 - 2(a^2 - b^2) x^2 + 6x^2y^2 - y^4 = 0.$$

Interchanging x and y, we get the equation corresponding to the y-axis.

Question 1099.

(K. J. Sanjana, M.A.):—(i) If n and r are positive integers, prove that

$${}_nC_{r-1} - \tfrac{1}{2} \cdot {}_nC_{r-2} + \tfrac{1}{3} \cdot {}_nC_{r-3} - \dots \dots + (-1)^{r-1} \frac{1}{r} {}_nC_0$$

$$= {}_nC_r \left\{ \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \dots\dots + \frac{1}{n-r+1} \right\}$$

(ii) If r is a positive integer, and n any rational positive number, prove that

$$ {}_nP_{r-1} + \tfrac{1}{2}\cdot {}_nP_{r-2} + \tfrac{1}{3}\cdot {}_nP_{r-3} + \ldots\ldots + \frac{1}{r}\cdot {}_nP_0 $$

$$ = {}_nP_r \left\{ \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{n+r-1} \right\}, $$

where $ {}_nP_r = \dfrac{n(n+1)\ \ldots\ldots\ (n+r-1)}{\angle r} $ and $ {}_nP_0 = 1. $

Solution by Martyn M. Thomas, N. Sundaram, V. M. Gaitonde and V. Ramachandran.

First Part.

The left-hand series = coefft. of x^r in $(1+x)^n \times \log(1+x)$

$$ = \frac{1}{r} \text{ coefft. of } x^{r-1} \text{ in } \frac{d}{dx}\left\{ (1+x)^n \cdot \log(1+x) \right\} $$

$$ = \frac{1}{r} \text{ coefft. of } x^{r-1} \text{ in } \left\{ (1+x)^{n-1} + n\,(1+x)^{n-1} \log(1+x) \right\} $$

$$ = \frac{1}{r}\cdot {}_{n-1}C_{r-1} + \frac{n}{r}\cdot \text{coefft. of } x^{r-1} \text{ in } (1+x)^{n-1}\log(1+x) $$

$$ = \frac{1}{r}\cdot {}_{n-1}C_{r-1} + \frac{n}{r}\cdot\frac{1}{r-1} \text{ coefft. of } x^{r-2} \text{ in } $$

$$ \left\{ (1+x)^{n-2} + (n-1)\,(1+x)^{n-2}\log(1+x) \right\} $$

$$ = \frac{1}{r}\cdot {}_{n-1}C_{r-1} + \frac{n}{r}\cdot\frac{1}{r-1}\cdot {}_{n-2}C_{r-2} + \frac{n(n-1)}{r(r-1)} \text{ coefft. of } x^{r-2} $$

$$ \text{in } (1+x)^{n-2} \log(1+x) $$

$$ = \frac{1}{n}\cdot {}_nC_r + \frac{1}{n-1}\cdot {}_nC_r \ \ldots\ldots + \frac{n(n-1)\ \ldots\ldots\ (n-r+2)}{r(r-1)\ \ldots\ldots\ 2} \text{ coefft. of } x \text{ in} $$

$$ (1+x)^{n-r+1} \log(1+x) $$

$$ = \frac{1}{n}\cdot {}_nC_r + \frac{1}{n-1}\cdot {}_nC_r + \ldots\ldots + \frac{1}{n-r+1}\cdot {}_nC_r. $$

Second Part.

The left-hand series = coefft. of x^r in $-\log(1-x) \times (1-x)^{-n}$

$$ = \frac{1}{r} \text{ coefft. of } x^{r-1} \text{ in } \frac{d}{dx}\left\{ -\log(1-x)\cdot(1-x)^{-n} \right\} $$

$$= \frac{1}{r} \text{ coefft. of } x^{r-1} \text{ in} \left\{ (1-x)^{-n-1} - n(1-x)^{-n-1} \log(1-x) \right\}$$

$$= \frac{1}{r} \cdot {}_{n+1}P_{r-1} + \frac{n}{r} \text{ coefft. of } x^{r-1} \text{ in}$$

$$\left\{ -\log(1-x) \cdot (1-x)^{-n-1} \right\}$$

$$= \frac{1}{r}\, {}_{n+1}P_{r-1} + \frac{n}{r} \cdot \frac{1}{r-1} \text{ coefft. of } x^{r-2} \text{ in}$$

$$\left\{ (1-x)^{-n-2} - (n+1)(1-x)^{-n-2} \cdot \log(1-x) \right\}$$

$$= \frac{1}{r} \cdot {}_{n+1}P_{r-1} + \frac{n}{r} \cdot \frac{1}{r-1} \cdot {}_{n+2}P_{r-2} + \frac{n(n+1)}{r(r-1)} \text{ coefft. of } x^{r-2}$$

$$\text{in} \left\{ -(1-x)^{-n-2} \log(1-x) \right\}$$

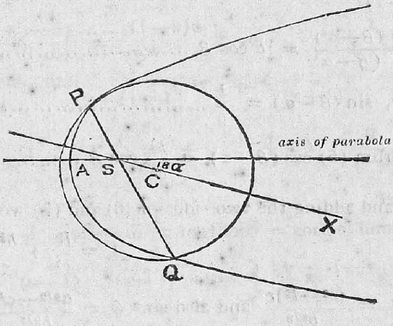
$$= \frac{1}{n} \cdot {}_{n}P_{r} + \frac{1}{n+1} \cdot {}_{n}P_{r} + \ldots\ldots + \frac{n(n+1)\ldots(n+r-2)}{r(r-1)\ \ldots\ldots 2} \text{ coefft. of } x \text{ in}$$

$$\left\{ -(1-x)^{-(n+r-1)} \cdot \log(1-x) \right\}$$

$$= \frac{1}{n} \cdot {}_{n}P_{r} + \frac{1}{n+1} \cdot {}_{n}P_{r} + \ldots\ldots + \frac{1}{n+r-1} \cdot {}_{n}P_{r}.$$

Question 1105.

(N. P. Pandya) :—S is a fixed point in the plane of a given circle. A variable parabola is described with S as focus to cut the circle in P and Q so that PSQ is always a straight line. Find the locus of the vertex o the parabola.



Solution by Giridhari Lal Gupta and C. Ranganatham.

Let S be taken for the pole and SC for the initial line where C is the centre of the given circle. Let the axis of the parabola make an angle α with the initial line. Let SC $= b$ and radius of the circle $= a$.

The equation of the parabola is $\frac{l}{r} = 1 - \cos(\theta - \alpha)$ ....(i)

And that of the circle is $r^2 - 2b\cos\theta . r + b^2 - a^2 = 0$ ....(2)

Let the equation of SP be $\theta = \beta$ ....(3)

and let SP and SQ taken positively be denoted by r_1 and r_2 respectively.

$\therefore$ From (2) and (3), r_1 and $-r_2$ are the roots of

$$r^2 - 2b\cos\beta . r + b^2 - a^2 = 0.$$

Whence $r_1 r_2 = a^2 - b^2$ and $r_1 - r_2 = 2b\cos\beta$ ....(A)

Again from (1) and (3) $\frac{l}{r_1} = 1 - \cos(\beta - \alpha) = 2\sin^2\frac{\beta - \alpha}{2}$

and $\frac{l}{r_2} = 1 - \cos(\pi + \beta - \alpha) = 1 + \cos(\beta - \alpha) = 2\cos^2\frac{\beta - \alpha}{2}$

$$\left.\begin{aligned} \therefore r_1 r_2 &= \frac{l^2}{4\sin^2\frac{\beta-\alpha}{2}\cos^2\frac{\beta-\alpha}{2}} = \frac{l^2}{\sin^2(\beta-\alpha)} \\ \text{and } r_1 - r_2 &= \frac{l}{2}\left(\frac{1}{\sin^2\frac{\beta-\alpha}{2}} - \frac{1}{\cos^2\frac{\beta-\alpha}{2}}\right) = \frac{2l\cos(\beta-\alpha)}{\sin^2(\beta-\alpha)} \end{aligned}\right\} \text{...... (B)}$$

Thus from relations (A) and (B) we have the following equations :—

$\frac{l^2}{\sin^2(\beta - \alpha)} = a^2 - b^2 = c^2$ (suppose) ....(4)

and $\frac{l\cos(\beta - \alpha)}{\sin^2(\beta - \alpha)} = b\cos\beta$ ....(5)

From (4), $\sin(\beta - \alpha) = \frac{l}{c}$ ....(6)

and $\therefore$ from (5) $\cos(\beta - \alpha) = \frac{bl}{c^2}\cos\beta$ ....(7)

Squaring and adding the two sides of (6) and (7) we get

$$1 = \frac{l^2}{c^2} + \frac{b^2 l^2}{c^4}\cos^2\beta$$

$\therefore \cos^2\beta = \frac{(c^2 - l^2)c^2}{b^2 l^2}$ and also $\sin^2\beta = \frac{a^2 l^2 - c^4}{b^2 l^2}$.

Now (4) may be written as $\frac{l^2}{c^2} = \{ \sin\beta \cos\alpha - \cos\beta \sin\alpha \}^2$

$$= \frac{(c^2-l^2)c^2}{b^2l^2}\sin^2\alpha + \frac{a^2l^2-c^4}{b^2l^2}\cos^2\alpha - \sin 2\alpha \,.\, c\,\frac{\sqrt{c^2-l^2}\,\sqrt{a^2l^2-c^4}}{b^2l^2}.$$

or $c\sqrt{c^2-l^2}\,\sqrt{a^2l^2-c^4}\,\sin 2\alpha = (a^2l^2-c^4)\cos^2\alpha$

$$+\,c^2(c^2-l^2)\sin^2\alpha - \frac{b^2l^4}{c^2} \quad \ldots \quad \ldots \quad (8)$$

Now the co-ordinates of the vertex are given by

$$\gamma = \frac{l}{2} \text{ and } \theta = \alpha + \pi.$$

Thus eliminating l and α we have the following equation as the locus :—

$$c\sqrt{c^2-4\gamma^2}\,\sqrt{4a^2\gamma^2-c^4}\,\sin^2\theta = (4a^2\gamma^2-c^4)\cos^2\theta$$
$$+\,c^2(c^2-4\gamma^2)\sin^2\theta - 16\,\frac{b^2}{c^2}\,\gamma^4.$$

Question 1107.

(CHARLES SALDANHA) :—If x be any odd integer not divisible by 3, prove that the integral part of $4^x - (2+\sqrt{2})^x$ is a multiple of 112.

Solution by K. J. Sanjana and N. Sundaram Iyer.

Lemma. Let $a+b+c=0$, and put $bc+ca+ab=s$ and $abc=t$.

Then $\left(1-\frac{a}{x}\right)\left(1-\frac{b}{x}\right)\left(1-\frac{c}{x}\right) = 1+\frac{s}{x^2}-\frac{t}{x^3}$

$$= 1-\left(\frac{t}{x^3}-\frac{s}{x^2}\right).$$

Take logarithms and equate the co-efficients of $1/x^n$; we get

$$\frac{1}{n}(a^n+b^n+c^n) = \text{co-efficient of } \frac{1}{x^n} \text{ in } \sum \frac{1}{r}\left(\frac{t}{x^3}-\frac{s}{x^2}\right)^r.$$

Let $n=6m+1$; the co-efficient of $1/x^n$ on the right side will occur in the terms from $\frac{1}{2m+1}\left(\frac{t}{x^3}-\frac{s}{x^2}\right)^{2m+1}$ to $\frac{1}{3m}\left(\frac{t}{x^3}-\frac{s}{x^2}\right)^{3m}$. It will be found to contain $t^{2m-1}.s^2,\ t^{2m-3}.s^5,\ \ldots\ t.s^{3m-1}$, which shows at once that it is divisible by s^2t, *i.e.*, by $abc(bc+ca+ab)^2$.

When $n = 6m - 1$, the co-efficient will contain factors ranging from $t^{2m-1} . s$ to $t . s^{3m-2}$, and is thus always divisible by st, *i.e.*, by

$$abc(bc + ca + ab).$$

Thus we deduce the theorem (which goes by the name of *Cauchy*) that when n is an odd number not divisible by 3, $a^n + b^n + c^n$ is divisible by $abc(bc + ca + ab)$ or $abc(bc + ca + ab)^2$ according as n is of the form $6m - 1$ or $6m + 1$, provided that $a + b + c$ vanishes.

As $(2 + \sqrt{2})^x + (2 - \sqrt{2})^x$ is an integer and $(2 - \sqrt{2})^x$ is a proper fraction, the integral part of $4^x - (2 + \sqrt{2})^x$ is $4^x - (2 + \sqrt{2})^x - (2 - \sqrt{2})^x$, which is seen to be $4^x + (-2 - \sqrt{2})^x + (-2 + \sqrt{2})^x$, as x is an odd integer. Now $4 + (-2 - \sqrt{2}) + (-2 + \sqrt{2}) = 0$ and x is not divisible by 3; hence by the Lemma the integral part is divisible by

$$(2 - 8 + 4\sqrt{2} - 8 - 4\sqrt{2}) . 4(2 - \sqrt{2})(-2 + \sqrt{2})$$

or by $$(2 - 8 + 4\sqrt{2} - 8 - 4\sqrt{2})^2 . 4(-2 - \sqrt{2})(-2 + \sqrt{2})$$

according as x is of the form $6m - 1$ or $6m + 1$; *i.e.*, by 112 or 112×14 numerically, and hence by 112 in the case proposed.

[*Note* :—The above is a solution with the aid of Cauchy's theorem. A solution without the aid of the above theorem is wanted. The question appears to be solvable without reference to Cauchy's theorem. It was taken from a Cambridge paper and set at the last B.A. Hon. Exam. of the Bombay University.]

Question 1108.

(COMMUNICATED) :—Prove that

$$\int_0^{\pi/2} \text{arc tan } (a\sqrt{\tan x})\, dx = \pi \text{ arc tan } (a\sqrt{2} + 1) - \frac{\pi^2}{4}.$$

Hence, or otherwise, prove that

$$\int_0^{\pi/2} \text{arc tan}\left[\frac{\sqrt{2 \tan x}}{1 + \tan x}\right] dx = \frac{\pi^2}{12}$$

and $$\int_0^{\pi/2} \text{arc tan}\left[\frac{2\sqrt{\tan x}}{1 + \tan x}\right] dx = \pi \text{ arc sin } \tfrac{1}{3}.$$

Solution by Martyn M. Thomas and several others.

Let $I = \int_0^{\pi/2} \arctan(a\sqrt{\tan x})\,dx.$

Then $\dfrac{dI}{da} = \displaystyle\int_0^{\pi/2} \frac{\sqrt{\tan x}}{1 + a^2\tan x}\,dx$

$$= \int_0^{\infty} \frac{2t^2dt}{(1 + a^2t^2)(1 + t^4)},\ \text{where } t^2 = \tan x.$$

$$= \int_0^{\infty} \left[\frac{2}{a^4 + 1}\left(\frac{-a^2}{1 + a^2t^2} + \frac{t^2 + a^2}{1 + t^4}\right)\right] dt,$$

resolving into partial fractions

$$= -\frac{2a}{a^4 + 1}\int_0^{\infty} \frac{d(at)}{1 + a^2t^2} + \frac{1}{a^4 + 1}\int_0^{\infty} \frac{2(t^2 + 1)}{t^4 + 1}\,dt + \frac{2(a^2 - 1)}{a^4 + 1}\int_0^{\infty} \frac{dt}{t^4 + 1}$$

$$= -\frac{2a}{a^4 + 1}\Big[\tan^{-1}(at)\Big]_0^{\infty} + \frac{1}{a^4 + 1}\Big[\sqrt{2}\tan^{-1}(t\sqrt{2} + 1) + \sqrt{2}\tan^{-1}(t\sqrt{2} - 1)\Big]_0^{\infty}$$

$$+ \frac{2(a^2 - 1)}{a^4 + 1}\left[\frac{1}{4\sqrt{2}}\log\left(\frac{t^2 + t\sqrt{2} + 1}{t^2 - t\sqrt{2} + 1}\right) + \frac{1}{2\sqrt{2}}\tan^{-1}(t\sqrt{2} + 1) + \frac{1}{2\sqrt{2}}\tan^{-1}(t\sqrt{2} - 1)\right]_0^{\infty}$$

$$= -\frac{2a}{a^4 + 1}\left[\frac{\pi}{2}\right] + \frac{1}{a^4 + 1}\left[\sqrt{2}\cdot\frac{\pi}{2} + \sqrt{2}\cdot\frac{\pi}{2}\right] + \frac{2(a^2 - 1)}{a^4 + 1}\left[\frac{1}{2\sqrt{2}}\cdot\frac{\pi}{2} + \frac{1}{2\sqrt{2}}\cdot\frac{\pi}{2}\right]$$

$$= \frac{\pi}{a^4 + 1}\left\{-a + \sqrt{2} + \frac{a^2 - 1}{\sqrt{2}}\right\} = \frac{\pi}{\sqrt{2}}\cdot\frac{a^2 - \sqrt{2}a + 1}{a^4 + 1}$$

$$= \frac{\pi}{\sqrt{2}}\cdot\frac{1}{a^2 + \sqrt{2}a + 1}.$$

$\therefore$ **Integrating with respect to** a,

$$I = \frac{\pi}{\sqrt{2}} \int \frac{da}{a^2 + \sqrt{2}a + 1} = \pi \tan^{-1}(a\sqrt{2} + 1) + C.$$

When $a = 0$, $I = 0$. $\therefore 0 = \pi \cdot \frac{\pi}{4} + C$. $\therefore C = -\frac{\pi^2}{4}$.

Hence $I = \pi \text{ arc tan } (a\sqrt{2} + 1) - \frac{\pi^2}{4}$ ... ... 1)

Second Part.

Now, $I = \int_0^{\pi/2} \tan^{-1}(a\sqrt{\tan x})\, dx.$

Also $I = \int_0^{\pi/2} \tan^{-1}[a\sqrt{\tan(\pi/2 - x)}]\, dx$

$$= \int_0^{\pi/2} \tan^{-1}(a\sqrt{\cot x})\, dx.$$

$$\therefore \quad 2I = \int_0^{\pi/2} \left[\tan^{-1}(a\sqrt{\tan x}) + \tan^{-1}(a\sqrt{\cot x})\right] dx$$

$$= \int_0^{\pi/2} \tan^{-1}\left[\frac{a(\sqrt{\tan x} + \sqrt{\cot x})}{1 - a^2}\right] dx$$

$$= \int_0^{\pi/2} \tan^{-1}\left[\frac{a}{1 - a^2} \cdot \frac{\sin x + \cos x}{\sqrt{\cos x \sin x}}\right] dx$$

$$= \int_0^{\pi/2} \tan^{-1}\left[\frac{a}{1 - a^2} \cdot \frac{1 + \tan x}{\sqrt{\tan x}}\right] dx$$

$$= \int_0^{\pi/2} \left[\frac{\pi}{2} - \cot^{-1}\left(\frac{a}{1 - a^2} \cdot \frac{1 + \tan x}{\sqrt{\tan x}}\right)\right] dx$$

$$= \int_0^{\pi/2} \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1 - a^2}{a} \cdot \frac{1 + \tan x}{\sqrt{\tan x}}\right)\right] dx.$$

$$\therefore \int_0^{\pi/2} \tan^{-1}\left(\frac{1 - a^2}{a} \cdot \frac{1 + \tan x}{\sqrt{\tan x}}\right) dx = \frac{\pi}{2} \cdot \frac{\pi}{2} - 2 . I$$

$$= \frac{3\pi^2}{4} - 2\pi \tan^{-1}(a\sqrt{2} + 1), \text{ from } (1) \quad \ldots \quad \ldots \quad (2)$$

Putting $\frac{1 - a^2}{a} = \sqrt{2}$ in the relation (2) obtained above, that is, $a = \frac{\sqrt{3} - 1}{\sqrt{2}}$,

$$\int_0^{\pi/2} \tan^{-1}\left(\frac{\sqrt{2} \tan x}{1 + \tan x}\right) dx = \frac{3\pi^2}{4} - 2\pi \tan^{-1}(\sqrt{3})$$

$$= \frac{3\pi^2}{4} - 2\pi\left(\frac{\pi}{3}\right) = \frac{\pi^2}{12}.$$

Again, putting $\frac{1 - a^2}{a} = 2$ in the relation (2), that is, $a = \sqrt{2} - 1$, we have

$$\int_0^{\pi/2} \tan^{-1}\left(\frac{2\sqrt{\tan x}}{1 + \tan x}\right) dx = \frac{3\pi^2}{4} - 2\pi \tan^{-1}(3 - \sqrt{2})$$

$$= \frac{3\pi^2}{4} - \pi \tan^{-1}\left(\frac{6 - 2\sqrt{2}}{6\sqrt{2} - 10}\right)$$

$$= \pi\left[\tan^{-1}(-1) - \tan^{-1}\left(\frac{3 - \sqrt{2}}{3\sqrt{2} - 5}\right)\right]$$

$$= \pi \tan^{-1}\left[\frac{2 - 2\sqrt{2}}{4\sqrt{2} - 8}\right]$$

$$= \pi \tan^{-1}\left(\frac{1}{2\sqrt{2}}\right) = \pi \sin^{-1}\left(\tfrac{1}{3}\right).$$

QUESTIONS FOR SOLUTIONS.

1151. (SELECTED) :—In any closed oval curve without singularities a normal chord PQ will have its maximum or minimum values, either when Q is the centre of curvature at P, or when PQ is normal at Q as well as at P, which must always be the case for two positions at least of PQ.

1152. (SELECTED):—Find the complete primitive of the differential equation

$$9xy^3 y_2 - 2 = 0. \qquad \text{(Forsyth : \textit{Diff. Eqns.})}$$

1153. (R. VAIDYANATHASWAMI) :—Find three collinear points on a cusped cubic, the tangents at which are concurrent. Prove that the line containing the three points passes through a fixed point and the point of concurrence of the tangents traces a fixed line.

1154. (R. VAIDYANATHASWAMI) :—Find all the conics which reciprocate a cusped cubic into itself.

1155. (R. VAIDYANATHASWAMI) :—A B C E are four points on a conic S. The polar line of ABC *w. r. t.* S, interests S in P Q and the polar line of E *w. r. t.* the triangle ABC intersects S in X, Y. Prove that the conic which is inscribed to ABC and XYP passes through E.

1156. (R VAIDYANATHASWAMI):—Shew that there are four conics inscribed to a △ ABC which pass through a given point E, and touch a given conic S passing through A B C E. Shew also that the four points of contact are such that the polar line of the △ formed by any three *w. r. t.* S passes through the fourth.

1157. (K. J SANJANA and K. B. MADHAVA) :—In all integral right-angled triangles in which the sides are prime to each other, the sums and the differences of the sides are (1) either primes every one of which has 2 for a quadratic residue or (2) the products of such primes and their powers.

1158. (N. DORAI RAJAN) :—If the sides of a triangle are in arithmetical progression prove that, with the usual notation, the straight line IG is parallel to one of the sides.

1159. (N. DORAI RAJAN) :—Is the following theorem true? : "The three cusped hypocycloid is the only quartic curve, (excluding a pair of concentric circles) in which all tangent chords are equal."

1160. (N. DORAI RAJAN) :—A Number N consists of p digits. It has the property that, when squared, it ends with the same digits in the same order as in itself. Find the number when (1) $p = 4$ and (2) $p = 5$.